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Cosmological constraints on primordial black holes produced in the near-critical gravitational collapse

Jun'ichi YOKOYAMA

Department of Physics, Stanford University, Stanford, CA 94305-4060 and
Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-01, Japan

Abstract

The mass function of primordial black holes created through the near-critical gravitational collapse is calculated in a manner fairly independent of the statistical distribution of underlying density fluctuation, assuming that it has a sharp peak on a specific scale. Comparing it with various cosmological constraints on their mass spectrum, some newly excluded range is found in the volume fraction of the region collapsing into black holes as a function of the horizon mass.

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In the study of gravitational collapse with formation of a black hole a critical phenomenon was discovered by Choptuik [1], who performed a series of numerical calculation of the evolution of a spherically symmetric system with a minimally coupled scalar field. He showed that the resultant mass of the black hole, if formed at all, scaled as $M_{\rm BH} \propto (p-p_c)^\gamma$ with $p \geq p_c$, where p is a control parameter of the system and p_c is the critical value above which a black hole forms. The remarkable feature of the above formula is that the power-law exponent was found to be independent of the choice of the control parameter. Since then a number of authors investigated various systems with scale-free matter ingredients and found essentially the same result with the unique value of the power index, $\gamma \simeq 0.36$ [2,3]. An analytic explanation has also been given by Koike et al. [4]. Although this is certainly an interesting subject of study in general relativity, its relevance to the black holes created in astrophysical or cosmological setting had not been known. Recently, however, Niemeyer and Jedamzik [5] observed the same type of the critical behavior in their new numerical calculation of the formation of primordial black holes (PBHs) [6,7] out of radiation fluid in the Friedmann-Robertson-Walker background.

PBHs are formed in the radiation dominated era when a perturbed region enters the Hubble radius if the amplitude of the fluctuation exceeds some critical value $\delta_c \sim 1/3$ [8], which has been confirmed by numerical analysis done two decades ago [9,10]. It has also been concluded that the resultant PBH has a mass of order of the horizon mass at formation. Consequently all the studies done so far to constrain the mass spectrum of PBHs are based on the assumption that PBHs of a specific mass is created at a specific epoch, namely, when the mass scale entered the Hubble radius as long as we are concerned with those formed during radiation domination [11]. The same assumption has also been made in building models of inflation to realize formation of PBHs on some specific mass scales [12–18].

According to Niemeyer and Jedamzik [5], however, PBHs are not only produced with about the horizon mass but also with much smaller masses with the scaling formula

$$M_{\rm BH} = K(\delta - \delta_c)^\gamma, \tag{1}$$

where as the control parameter δ they chose the additional mass in the perturbed region in unit of the horizon mass when the relevant scale entered the Hubble radius. They investigated two classes of configurations of the perturbed region and found a universal power index of $\gamma \cong 0.35$ in agreement with Evans and Coleman [3] and Koike etal. [4]. They have also confirmed it is independent of the choice of the control parameter [5], as it should be.

Although the scaling relation (1) is expected to be valid only in the immediate neighborhood of δ_c, most black holes are expected to form with such an initial value of δ, because it has generically a rapidly declining probability distribution function (PDF) near $\delta = \delta_c$ and the probability to find $\delta \gg \delta_c$ is exponentially smaller. Hence it is sensible to calculate the expected mass function of PBHs using the formula (1). The above-mentioned feature of the PDF also allows us to estimate the mass function fairly independent of the specific form of the PDF of primordial density or curvature fluctuations as seen below.

The purpose of the present paper is to investigate the effect of the near-critical gravitational collapse on the cosmological constraints on the mass spectrum of PBHs and on the model building

of inflation to produce PBHs. As a result we find that, thanks to the fact that γ is relatively small, we do not have to change the conventional view to the issue drastically except that we find a new constraint on the fraction of the space collapsing to PBHs when the horizon mass is in the range between 4×10^{14}g and 6×10^{16}g.

In this paper we study the case of a primordial spectrum of density fluctuation that is sharply peaked on a single but arbitrary mass scale just as in the initial configuration adopted in [5]. Several models have been proposed to realize such a spectral shape in inflationary cosmology [12,15–18]. Even in the case with such a simple spectrum, the configuration of a perturbed region has functional degrees of freedom and one must in principle calculate both the probability to realize each initial configuration and the resultant mass of the black hole, if formed at all, in order to calculate the mass function of PBHs. In the language of the near-critical collapse, there are many possible one-parameter families of the initial data in the configuration functional space to approach the critical surface and we find different values of δ_c and K for each family with the exponent γ being the only universal quantity. Hence we should first calculate the mass function of PBHs when the initial configuration is changed along each one-parameter family near the critical surface and then integrate over such families in the functional space together with their relative probability to obtain the overall mass function, which is a formidable task. Fortunately, however, there is yet another common feature in PBH formation, which is the fact that the typical mass of the black hole is of order of the horizon mass independent of the shape of the perturbed region even in the presence of the critical phenomenon as shown explicitly in [5]. We thus reduce the problem with infinitely many degrees of freedom to a one-dimensional issue, that is, we adopt an approximation that the system is governed by a single parameter, δ, characterizing the amplitude of fluctuations, and that the most common black hole has the horizon mass, M_H, when the peak of fluctuation entered the Hubble radius at $t = t_H$.

Since the PDF of δ, $P(\delta)$, is a steeply declining function around δ_c we write it as

$$P(\delta)d\delta = e^{-f(\delta)}d\delta, \tag{2}$$

where $f(\delta)$ is a well-behaved function around $\delta \simeq \delta_c$. If δ is Gaussian distributed, as assumed in most literature [8,19,20], $f(\delta)$ is explicitly given by

$$f(\delta) = \ln(\sqrt{2\pi}\sigma) + \frac{\delta^2}{2\sigma^2}, \tag{3}$$

with σ being the dispersion of δ.

Then the probability, $\beta(M_H)$, that the relevant mass scale has an above-threshold amplitude of fluctuations to collapse into a black hole as it enters the Hubble radius is given by

$$\beta(M_H) = \int_{\delta_c} P(\delta)d\delta = \int_{\delta_c} e^{-f(\delta)}d\delta, \tag{4}$$

which is also equal to the volume fraction of the region collapsing to a black hole at $t = t_H$. Due to the fact that $P(\delta)$ is a steeply decreasing function the integral is sensitive to only its lower bound δ_c. Furthermore we can Taylor expand $f(\delta)$ as

$$f(\delta) = f(\delta_c) + f'(\delta_c)(\delta - \delta_c) + \cdots \equiv f_c + q(\delta - \delta_c) + \cdots, \tag{5}$$

to find

$$\beta(M_H) \cong \frac{1}{q} e^{-f_c}. \tag{6}$$

The above formula (6) based on the approximation (5) is applicable if $|f''(\delta_c)| \ll q^2$. In the case of a Gaussian distribution this corresponds to $\sigma \ll \delta_c$. Putting $\delta_c = 1/3$ and using (3) in (6) we recover Carr's formula [8], $\beta \simeq \sigma \exp[-1/(18\sigma^2)]$. If $P(\delta)$ is non-Gaussian, we must analyze case by case, but as long as we consider the case with small enough $\beta(M_H)$ in the cosmologically allowed range [11] and use specific non-Gaussian distributions reported in the literature [16,18,21], we may justify (5). Hence this simple approximation enjoys a wide applicability.

Then the contribution of PBHs to the density parameter at formation $t_f \gtrsim t_H$, is given by

$$\Omega_{\rm BH}(t_f) = \frac{1}{M_H}\int_{\delta_c} M_{\rm BH}(\delta)P(\delta)d\delta = \frac{1}{M_H}\int_0^\infty MP[\delta(M)]\frac{1}{\gamma}\left(\frac{M}{K}\right)^{\frac{1}{\gamma}}\frac{dM}{M}, \tag{7}$$

with $\delta(M) = \delta_c + (M/K)^{\frac{1}{\gamma}}$. Using (2) and (5) in the above integral, the differential mass spectrum reads

$$\frac{d\Omega_{\rm BH}(M,t_f)}{d\ln M} = \frac{M}{\gamma M_H}\left(\frac{M}{K}\right)^{\frac{1}{\gamma}} \exp\left[-f_c - q\left(\frac{M}{K}\right)^{\frac{1}{\gamma}}\right], \tag{8}$$

which has a peak at $M_{\rm max} \equiv Kq^{-\gamma}(1+\gamma)^\gamma$. Since the typical black hole mass is around M_H as stated above, let us identify $M_{\rm max}$ with M_H following the line of thought explained above. We then find

$$\begin{aligned}\frac{d\Omega_{\rm BH}(M,t_f)}{d\ln M} &= \beta(M_H)\left(1+\frac{1}{\gamma}\right)\left(\frac{M}{M_H}\right)^{1+\frac{1}{\gamma}} \exp\left[-(1+\gamma)\left(\frac{M}{M_H}\right)^{\frac{1}{\gamma}}\right], \\ &\cong 3.86\beta(M_H)\left(\frac{M}{M_H}\right)^{3.86} \exp\left[-1.35\left(\frac{M}{M_H}\right)^{2.86}\right],\end{aligned} \tag{9}$$

which is now independent of q and K. Formal integration of (9) from $M = 0$ to ∞ yields $\Omega_{\rm BH}(t_f) \simeq (1+\gamma)^{-1}\Gamma(\gamma)\beta(M_H) = 0.80\beta(M_H)$ for $\gamma = 0.35$, as opposed to the case all the PBHs have the horizon mass where we would find $\Omega_{\rm BH}(t_f) = \beta(M_H)$. Thus the total mass density of PBHs does not change practically even if we take the effect of the near-critical collapse into account.

The mass spectrum (9) is depicted in Fig. 1 where we find that the abundance of smaller-mass black holes are suppressed even in the presence of the critical behavior. We can hence convince ourselves that the previous assumption that only horizon-mass black holes are likely be produced is a good approximation in model building where we try to identify some astrophysical objects such as MACHOs [22] with PBHs [15,17,18]. Indeed, if too many Jupiter-mass PBHs were produced simultaneously in an attempt to produce MACHO-mass PBHs, PBH-explanation of MACHOs would be ruled out because the observation of the MACHO group is very sensitive to Jupiter-mass objects and their abundance in the galactic halo has already been stringently constrained [23]. However, the mass fraction (9) implies we are free from such a problem.

FIGURES

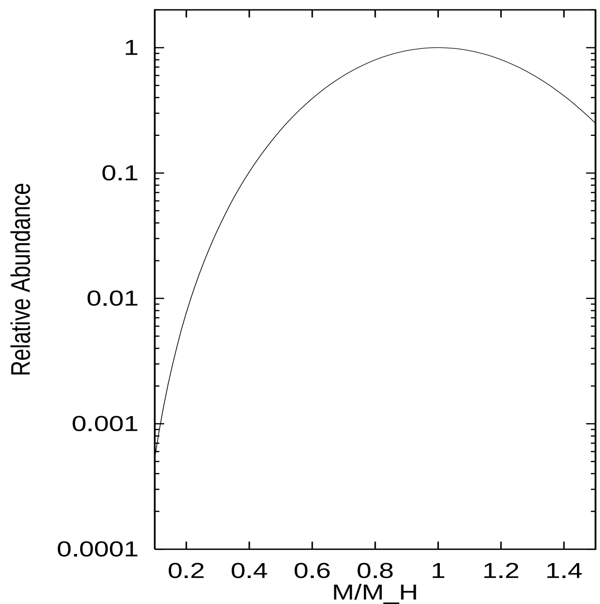

FIG. 1. Expected mass function of PBHs (9) created through the near-critical gravitational collapse in the case density fluctuation has a sharp peak on the mass scale M_H.

We must, on the other hand, consider its effect on cosmological constraints more carefully. At later time t we find the mass spectrum, $\mathcal{F}(M,t;t_f)$, of the holes produced at t_f as

$$\begin{aligned}\mathcal{F}(M,t;t_f) \equiv \frac{d\Omega_{\rm BH}(M,t;t_f)}{d\ln M} &= \frac{d\Omega_{\rm BH}(M,t_f)}{d\ln M}\left(\frac{t}{t_f}\right)^{\frac{1}{2}}, \qquad t < t_{eq}, \\ &= \frac{d\Omega_{\rm BH}(M,t_f)}{d\ln M}\left(\frac{t_{eq}}{t_f}\right)^{\frac{1}{2}}, \qquad t > t_{eq}, \end{aligned} \tag{10}$$

where

$$t_{eq} = 4.2\times 10^{10}(\Omega_0 h^2)^{-2}{\rm sec} = 1.9\times 10^{12}\Gamma^{-2}{\rm sec}, \qquad \Gamma \equiv \frac{\Omega_0}{0.3}\left(\frac{h}{0.7}\right)^2 = \frac{\Omega_0 h^2}{0.147}, \tag{11}$$

is the equality time with Ω_0 and h being current values of the total density parameter and the Hubble parameter in unit of 100km/sec/Mpc, respectively.

Cosmological constraints on the mass spectrum of the PBHs can be classified to three classes. The first one applies to heavy black holes with mass $M_{\rm BH} > 4\times 10^{14}$g which have not evaporated by now. Current mass density of such holes should not exceed the total mass density of the universe. The second class is due to the radiation of high energy particles from evaporating black holes [24,25]. Various constraints have been imposed from primordial nucleosynthesis [26], microwave background radiation [27], and gamma-ray background radiation [28]. Finally, there

may be yet another class of constraints if PBHs do not evaporate completely but leave relics with mass of order of the Planck mass or larger [29]. Their mass density should remain small enough.

Even if we adopt the new mass spectrum (1), cosmological constraints of the first and the third classes described above are not altered practically because the total number of black holes as well as the abundance of most typical black holes, which have approximately the horizon mass at formation and dominates their mass density, are not expected to change significantly as shown above. In particular, the constraint of the first class, namely, that from the total mass density of PBHs reads

$$\beta(M_H) < 2 \times 10^{-19} \Gamma \left(\frac{M_H}{4 \times 10^{14} \mathrm{g}} \right)^{\frac{1}{2}} \frac{\Omega_0}{0.3}, \tag{12}$$

demanding that $\Omega_{\rm BH}$ today should not exceed Ω_0.

We now concentrate on the constraints associated with evaporation. The lifetime of a black hole with mass M is given by

$$t_{ev}(M) = 1.7 \times 10^3 \tilde{g}^{-1} \left(\frac{M}{M_{Pl}} \right)^3 t_{Pl}, \tag{13}$$

where M_{Pl} (t_{Pl}) is the Planck mass (time) and $\tilde{g}$ is the effective number of massless state radiated from the black hole in units of 7.25 which applies at the low-energy or large mass limit [20]. A black hole will evaporate before the equality time if its mass satisfies the inequality $M < 6 \times 10^{12} \tilde{g}^{\frac{1}{3}} \Gamma^{-\frac{2}{3}} \mathrm{g} \equiv M_{eq}$. Eliminating t_f from both sides of (10) by virtue of the relation $M_H = M_{Pl}^2 t_H \simeq M_{Pl}^2 t_f$, we find

$$\begin{aligned} \mathcal{F}(M, t_{ev}(M); M_H) &= 1.6 \times 10^2 \tilde{g}^{-\frac{1}{2}} \beta(M_H) \left(\frac{M}{M_H} \right)^{4.36} \frac{M}{M_{Pl}} \exp \left[-1.35 \left(\frac{M}{M_H} \right)^{2.86} \right], \ M < M_{eq} \quad (14) \\ &= 2.3 \times 10^{28} \Gamma^{-1} \beta(M_H) \left(\frac{M}{M_H} \right)^{3.86} \left(\frac{M_{Pl}}{M_H} \right)^{\frac{1}{2}} \exp \left[-1.35 \left(\frac{M}{M_H} \right)^{2.86} \right], \ M > M_{eq}. \end{aligned}$$

The above mass spectrum should be compared with the following constraints of the second class [11,19,20,25–28].

$$\begin{aligned} \mathcal{F}(M, t_{ev}(M)) &\lesssim 1 \times 10^{-2} M_{10}^{\frac{1}{2}}, && \text{for } M = 10^9 - 10^{10} \mathrm{g}, \\ &\lesssim 5 \times 10^{-7} M_{10}^{\frac{3}{2}}, && \text{for } M = 10^{10} - 10^{11} \mathrm{g}, \\ &\lesssim 2 \times 10^{-8} M_{10}^{\frac{7}{2}}, && \text{for } M = 10^{11} - 6 \times 10^{11} \mathrm{g}, \\ &\lesssim \quad 4 \times 10^{-2}, && \text{for } M = 6 \times 10^{11} - 10^{13} \mathrm{g}, \\ &\lesssim \quad 1 \times 10^{-8}, && \text{for } M \simeq 4 \times 10^{14} \mathrm{g}, \end{aligned} \tag{15}$$

where $M_{10} \equiv M/10^{10}$ g.

Putting $M = M_H$ in (14) and comparing it with (15), we practically recover the constraints on $\beta(M_H)$ previously obtained in the literature, which are depicted by a solid line in Fig. 2. In the presence of the near-critical collapse, however, we should also consider constraints on $\beta(M_H)$

arising from PBHs with mass $M \ll M_H$. In fact, however, due to the steep shape of (14) we find a newly excluded range [1] only in the region $M_H > 4 \times 10^{14}$g, which is imposed from the last constraint of (15), namely,

$$\beta(M_H) < 2 \times 10^{-27}\Gamma \left(\frac{M_H}{4 \times 10^{14}\text{g}}\right)^{4.36} \exp\left[1.35\left(\frac{M_H}{4 \times 10^{14}\text{g}}\right)^{-2.86}\right]. \quad (16)$$

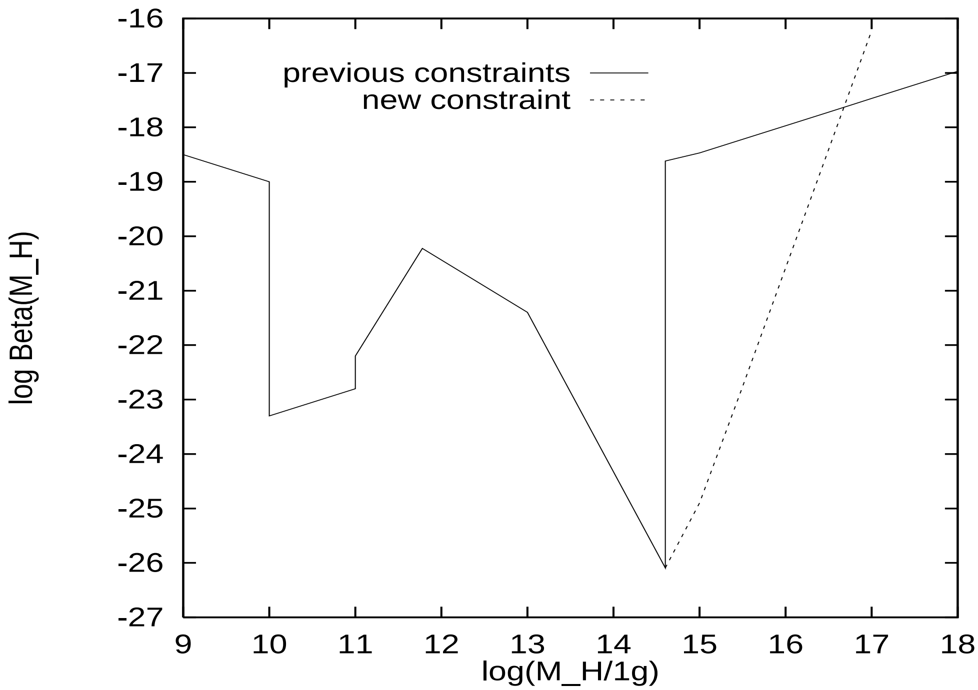

FIG. 2. Cosmological constraints on the volume fraction of the region, β, above the threshold of PBH formation when the horizon mass is equal to M_H. The region between the solid and the dotted lines is newly excluded.

As is seen in Fig. 2, this new constraint is more stringent than (12) for $M_H < 6 \times 10^{16}$g.

It has been argued that high energy phenomenon associated with currently evaporating black holes with mass $\sim 4 \times 10^{14}$g can explain the origin of a class of gamma-ray burst [30]. If this is the case, their abundance should be around $\Omega_{\rm BH} = 10^{-8}$ today. If such a tiny amount of black holes were created at the low-mass tail of the near-critical collapse, one could explain the origin of both such bursts and dark matter simultaneously. That is, PBHs with mass around 6×10^{16}g could be the dominant dark matter component which makes up $\Omega \simeq 0.3$ today.

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[1]There may be another small newly excluded region around $M_H \gtrsim 10^{11}$g. However, we did not depict it in Fig. 2 since it is based on a rather qualitative constraint from nucleosynthesis obtained two decades ago, and refined calculations may well alter it.

In summary, we have obtained an approximate but generic form of the mass function of primordial black holes which are produced through near-critical gravitational collapse of radiation fluid in the early universe in the case density fluctuations have a sharp peak on a specific scale. It is fairly independent of the statistical distribution of density fluctuations. Due to the smallness of the critical exponent $\gamma = 0.35$ the resultant mass function has a steep spectrum in the low-mass tail, so that the previous assumption that the PBHs are created with nearly the horizon mass at formation is basically correct. Nevertheless we find some newly excluded range in the volume fraction of the region with $\delta > \delta_c$ as a function of the horizon mass as depicted in Fig. 2.

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